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Lines of curvature on surfaces immersed in $\mathbb{R}^4$

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—Dedicated to the memory of Ricardo Mañé.

Abstract. The differential equation of the *lines of curvature* for immersions of surfaces into $\mathbb{R}^4$ is established. It is shown that, for a class of generic immersions of a surface into $\mathbb{R}^4$ in the C^r-topology, $r \geq 4$, all of the umbilic points are locally topologically stable. This type of umbilic points is described.

1. Introduction

This article is devoted to the study of the possible configurations of the lines of principal curvature around umbilic points on surfaces which are immersed in $\mathbb{R}^4$. We classify the locally topologically stable umbilic points and show that they appear generically.

The notion of principal direction for a smooth immersion $f : M \to \mathbb{R}^4$ which we use and introduce is due to J.A. Little ([Lit]), and is an extension of the classical three-dimensional concept (see [R-S] for another possible extension). A *principal direction at* p is a line in T_pM generated by a unitary vector which makes extremal the length of $\alpha(X, X)$, where α is the second fundamental form of the immersion f at p and X varies on the unitary circle in T_pM. The set $\mathcal{E}$ of values of $\alpha(X, X)$ is an ellipse, called ellipse of curvature, which can degenerate into a line segment, a circle or a point. Also, it is easily seen that as X goes once around

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the circle, $\alpha(X,X)$ goes twice around around $\mathcal{E}$. Therefore, when $\mathcal{E}$ is either an ellipse or a line segment, there are four principal direcctions at p; when $\mathcal{E}$ is either a circle or a point, we say p is a *umbilic point of* f. The principal lines of curvature of α are those curves in M, disjoint from umbilic points, which are tangent to principal lines.

The differential equation of the principal lines of curvature is established in paragraph two. In paragraph three we study a class of generic umbilic points called simple. Assuming that all umbilics of M are simple and that $P : PM \to M$ is the projective bundle, in paragraph four we show that the set of points (p, L), where $p \in M$ and $L \in T_pM$ is a principal direction, constitutes a smooth two-submanifold LM of PM which carries information that is needed in paragraph five to prove our chief result:

Theorem 1.1. *Under generic conditions and if* $f : M \to \mathbb{R}^4$ *is a smooth immersion of a surface* M *and* $p \in M$ *is a umbilic point of* f, *then there are isothermic coordinates* $(u,v) : (M,p) \to (\mathbb{R}^2, 0)$ *such that the differential equation of the principal lines of* f *in these coordinates is of the form*

$$4(Au+Bv+S(u,v))(du^2-dv^2)dudv+(v+R(u,v))(du^4-6du^2dv^2+dv^4) = 0$$

where $A \neq 0$ *and* B *are real numbers, and* $S(u,v)$ *and* $R(u,v)$ *are real valued functions which satisfy*

$$S(0,0) = R(0,0) = \frac{\partial S}{\partial u}(0,0) = \frac{\partial S}{\partial v}(0,0) = \frac{\partial R}{\partial u}(0,0) = \frac{\partial R}{\partial v}(0,0) = 0.$$

Moreover, under any one of the following conditions, the umbilic point p *is locally topologically stable and its phase portrait is obtained by making into one (by a rigid translation) the pair of pictures (nets) of the indicated figure:*

(a) *Condition* H3 *(Fig.1):* $\Delta < 0$,

(b) *Condition* H4 *(Fig.2):* $\Delta > 0$, $A < 0$ *and* $A \neq -1/4$,

(c) *Condition* H5 *(Fig.3):* $\Delta > 0$, $A > 0$, *where*

$$\Delta = 16[4(1+B^2)^3 + 24(1+B^2)^2 A + 8(5-B^2)(1+B^2)A^2 + \\ + 4(9+B^2)A^3 + (17+B^2)A^4 + 4A^5] .$$

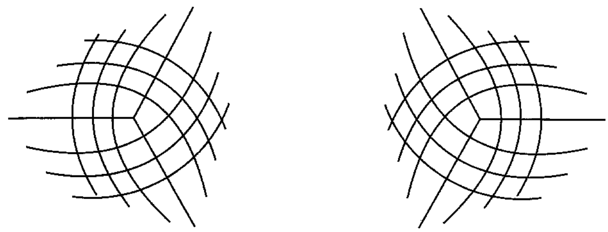

Figure 1.

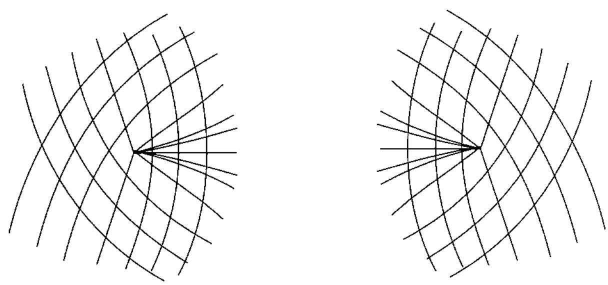

Figure 2.

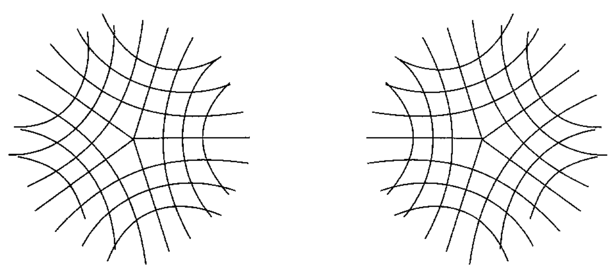

Figure 3.

Concerning Theorem 1.1 we may remark:

(a) $A \neq 0$ is a transversality (generic) condition which characterizes a simple umbilic point;

(b) Within the considered coordinates, the umbilic point p of type Hi, $i = 3, 4, 5,$ has i separatrices whose slopes at the origin are the roots of a polynomial having Δ as its discriminant;

(c) It will be shown that the principal lines around the umbilic p of Theorem 1.1 (which satisfies either of the conditions H3 or H4 or else H5) make up two pairwise transversal nets $\mathcal{F}_1$ and $\mathcal{F}_2$. We say that p is *locally topologically stable* when both $\mathcal{F}_1$ and $\mathcal{F}_2$ are locally topologically stable, around p. This definition for nets is similar to that for the case of principal lines of surfaces immersed in $\mathbb{R}^3$, around an isolated umbilic point, which can be seen in [G-S].

2. Differential equation of the lines of curvature

Let $f : M \to \mathbb{R}^4$ be a smooth immersion of a surface M. Let $U \subset M$ be an open neighborhood with isothermic coordinates (u, v). Let $z = u+iv$, and let $\lambda = |\partial_u| = |\partial_v|$ where $\partial_u = \dfrac{\partial}{\partial u}$ and $\partial_v = \dfrac{\partial}{\partial v}$.

We introduce the two Wirtingen operators

$$\partial_z = \frac{1}{\sqrt{2}}(\partial_u - i\partial_v) \qquad \text{and} \qquad \partial_{\bar{z}} = \frac{1}{\sqrt{2}}(\partial_u + i\partial_v), \tag{1}$$

and denote

$$\begin{aligned} \sigma &= \alpha(\partial_z, \partial_z), & \tau &= \alpha(\partial_z, \partial_{\bar{z}}) \\ a &= \mathrm{Re}(\langle \sigma, \sigma \rangle), & b &= 2\mathrm{Im}(\langle \sigma, \sigma \rangle) \end{aligned} \tag{2}$$

where $\langle\ ,\rangle$ is a bilinear complex extension of the inner product of $T^{\perp}M$ to $T^{\perp}M \otimes \mathbb{C}$, with $T^{\perp}M$ denoting the normal bundle.

This paragraph is devoted to the proof of the following result:

Theorem 2.1. *Let $f : M \to \mathbb{R}^4$ be a smooth immersion of a surface M. In isothermic coordinates $(u, v) : (M, p) \to (\mathbb{R}^2, 0)$, the differential equation of the lines of curvature of f is given by*

$$4a(u, v)(du^2 - dv^2)dudv + b(u, v)(du^4 - 6du^2dv^2 + dv^4) = 0 \tag{3}$$

where $a = a(u, v)$ and $b = b(u, v)$ are the real valued functions of (2). Moreover, p is a umbilic point if and only if $a(0, 0) = b(0, 0) = 0$.

Conversely, for any given analytic functions $a, b : U \to \mathbb{R}$ defined on an open neighborhood $U \subset \mathbb{R}^2$ of a point p, there exists an immersion $f : V \to \mathbb{R}^4$ where $V \subset U$ is some small open neighborhood of p such that the differential equation of the lines of curvature of f is given by (3) and that the coordinates (u, v) are isothermic.

To prove the theorem we need the next result.

Lemma 2.2. *Suppose the assumptions of above. Let* $\{e_3, e_4\}$ *be a normal frame. Let* $\eta = \eta(u,v)$ *be a smooth function such that*

$$\begin{aligned} \nabla^{\perp}_{\partial_u} e_3 &= \eta e_4, & \nabla^{\perp}_{\partial_u} e_4 &= -\eta e_3 \\ \nabla^{\perp}_{\partial_v} e_3 &= 0, & \nabla^{\perp}_{\partial_v} e_4 &= 0. \end{aligned} \tag{4}$$

If we denote

$$\sigma_{\beta 1} = \mathrm{Re}(\langle \sigma, e_\beta \rangle), \quad \sigma_{\beta 2} = \mathrm{Im}(\langle \sigma, e_\beta \rangle), \quad \tau_\beta = \langle \tau, e_\beta \rangle, \quad \beta = 3, 4, \tag{5}$$

then the Gauss, Ricci and Codazzi equations may be written, respectively, as

$$\lambda_{vv} = \frac{1}{\lambda}(-\sigma_{31}^2 - \sigma_{32}^2 - \sigma_{41}^2 - \sigma_{42}^2 + \tau_3^2 + \tau_4^2 + \lambda_u^2 + \lambda_v^2 - \lambda\lambda_{uu}) \tag{6}$$

$$(\eta)_v = \frac{2}{\lambda^2}(\sigma_{41}\sigma_{32} - \sigma_{31}\sigma_{42}) \tag{7}$$

$$\begin{aligned} (\sigma_{32})_v &= (\sigma_{31})_u - (\tau_3)_u - \eta\sigma_{41} + \eta\tau_4 + \frac{2}{\lambda}\lambda_u\tau_3 \\ (\sigma_{31} + \tau_3)_v &= -(\sigma_{32})_u + \eta\sigma_{42} + \frac{2}{\lambda}\lambda_v\tau_3 \\ (\sigma_{42})_v &= (\sigma_{41})_u - (\tau_4)_u + \eta\sigma_{31} - \eta\tau_3 + \frac{2}{\lambda}\lambda_u\tau_4 \\ (\sigma_{41} + \tau_4)_v &= -(\sigma_{42})_u - \eta\sigma_{32} + \frac{2}{\lambda}\lambda_v\tau_4 \end{aligned} \tag{8}$$

Proof. We use notations (1) and (2). Let us first consider the Gauss equation

$$\langle R(\partial_z, \partial_{\bar z})\partial_z, \partial_{\bar z}\rangle = \langle \alpha(\partial_z, \partial_z), \alpha(\partial_{\bar z}, \partial_{\bar z})\rangle - |\alpha(\partial_z, \partial_{\bar z})|^2. \tag{9}$$

Note that

$$\begin{aligned} R(\partial_z, \partial_{\bar z})\partial_z &= \nabla_{\partial_z}\nabla_{\partial_{\bar z}}\partial_z - \nabla_{\partial_{\bar z}}\nabla_{\partial_z}\partial_z \\ &= -\nabla_{\partial_{\bar z}}\left(\frac{2}{\lambda}\frac{\partial\lambda}{\partial z}\right)\partial_z \\ &= -\nabla_{\partial_{\bar z}}\left(2\frac{\partial \log\lambda}{\partial z}\right)\partial_z \\ &= -\Delta \log\lambda\,\partial_z \end{aligned}$$

which implies

$$\langle R(\partial_z, \partial_{\bar z})\partial_z, \partial_{\bar z}\rangle = -\lambda^2\Delta\log\lambda.$$

Hence the Gauss equation has the form

$$-\lambda^2 \Delta \log \lambda = |\sigma|^2 - |\tau|^2$$

and may be rewritten as (6).

We now consider the Ricci equation

$$R^\perp(\partial_z, \partial_{\bar z})v = \alpha\sigma_v\partial_{\bar z}, \partial_z) - \alpha(\sigma_v\partial_z, \partial_{\bar z}), \ v \in T^\perp M. \tag{10}$$

Note that

$$\begin{aligned} R^\perp(\partial_z, \partial_{\bar z})e_3 &= \alpha(\sigma_{e_3}\partial_{\bar z}, \partial_z) - \alpha(\sigma_{e_3}\partial_z, \partial_{\bar z}) \\ &= \frac{1}{\lambda^2}\Big[\alpha(\overline{\sigma}_3\partial_z + \tau_3\partial_{\bar z}, \partial_z) + \alpha(\sigma_3\partial_{\bar z} + \tau_3\partial_z, \partial_z)\Big] \\ &= \frac{1}{\lambda^2}(\overline{\sigma}_3\sigma - \sigma_3\overline{\sigma}) \\ &= 2\frac{i}{\lambda^2}\operatorname{Im}(\overline{\sigma}_3\sigma), \end{aligned}$$

hence

$$\langle R^\perp(\partial_z, \partial_{\bar z})e_3, e_4\rangle = 2\frac{i}{\lambda^2}\operatorname{Im}(\overline{\sigma}_3\sigma_4).$$

We also obtain that

$$\begin{aligned} R^\perp(\partial_z, \partial_{\bar z})e_3 &= \nabla^\perp_{\partial_z}\nabla^\perp_{\partial_{\bar z}}e_3 - \nabla^\perp_{\partial_{\bar z}}\nabla^\perp_{\partial_z}e_3 \\ &= \nabla^\perp_{\partial_z}\left(\frac{\eta}{\sqrt{2}}e_4\right) - \nabla^\perp_{\partial_{\bar z}}\left(\frac{\eta}{\sqrt{2}}e_4\right) \\ &= \frac{1}{\sqrt{2}}\left(\frac{\partial\eta}{\partial z} - \frac{\partial\eta}{\partial\bar z}\right)e_4 \\ &= -i\frac{\partial\eta}{\partial v}\ e_4\ , \end{aligned}$$

and thus

$$\langle R^\perp(\partial_z, \partial_{\bar z})e_3, e_4\rangle = -i\frac{\partial\eta}{\partial v}.$$

This implies that the Ricci equation has the form

$$\frac{\partial\eta}{\partial v} = -\frac{2}{\lambda^2}\operatorname{Im}(\overline{\sigma}_3\sigma_4)$$

which may be rewritten as (7).

Finally, we consider the Codazzi equation

$$(\nabla^\perp_{\partial_z}\alpha)(\partial_{\bar z}, \partial_z) = (\nabla^\perp_{\partial_{\bar z}}\alpha)(\partial_z, \partial_z). \tag{11}$$

We have

$$\nabla^{\perp}_{\partial_{\bar z}}\sigma = \nabla^{\perp}_{\partial_z}\tau - 2\frac{\partial \log \lambda}{\partial z}\tau.$$

Also, we find that

$$\nabla^{\perp}_{\partial_{\bar z}}\sigma = \left(\frac{\partial \sigma_3}{\partial \bar z} - \sigma_4\eta\right)e_3 + \left(\frac{\partial \sigma_4}{\partial \bar z} + \sigma_3\eta\right)e_4$$

and that

$$\nabla^{\perp}_{\partial_z}\tau = \left(\frac{\partial \tau_3}{\partial z} - \tau_4\eta\right)e_3 + \left(\frac{\partial \tau_4}{\partial z} + \tau_3\eta\right)e_4.$$

Hence

$$\begin{aligned}\frac{\partial \sigma_3}{\partial \bar z} - \sigma_4\eta &= \frac{\partial \tau_3}{\partial z} - \tau_4\eta - 2\frac{\partial \log \lambda}{\partial z}\tau_3\\ \frac{\partial \sigma_4}{\partial \bar z} + \sigma_3\eta &= \frac{\partial \tau_4}{\partial z} + \tau_3\eta - 2\frac{\partial \log \lambda}{\partial z}\tau_4\end{aligned}$$

which may be rewritten as (8) □

Proof of Theorem 2.1

The differential equation of the lines of curvature of f is given by $\mathrm{Im}(\langle\sigma,\sigma\rangle dz^4) = 0$ ([GGST, Prop. 5.1, pp.103]) which is equivalent to (3) and thus we have the first statement.

For the second, we need to prove that, for any given local analytic functions $a, b : (U,p) \to \mathbb{R}$ as in the assumptions, there exists a local analytic immersion f such that the differential equation of the lines of curvature of f is given by (3) and that the coordinates (u,v) are isothermic.

If we find a solution $\lambda > 0$, η, σ_{31}, σ_{32}, σ_{41}, $\sigma_{42}, \tau_3, \tau_4$ of system (6) – (8) of Lemma 2.2 such that each one of these functions is defined in an open neighborhood $V \subset U$ of p, then the theorem of existence and unicity of immersions [Jac] guarantees the existence of a local immersion $f : V \to \mathbb{R}^4$ which has $\lambda^2 = E = G$ and $F = 0$ as coefficients of its first fundamental form. On the other hand, if this solution satisfies the system

$$a = \sigma_{31}^2 - \sigma_{32}^2 + \sigma_{41}^2 - \sigma_{42}^2, \qquad b = 2(\sigma_{31}\sigma_{32} + \sigma_{41}\sigma_{42})\ , \tag{13}$$

then the differential equation of the principal lines of curvature is given by (3) and thus the proof of the theorem will follow.

For this we first define $\Lambda_i = \Lambda_i(u, v, \sigma_{32}, \sigma_{42})$, $i = 3, 4$, by

$$\Lambda_3 = \frac{b\sigma_{32} + \sigma_{42}c}{2(\sigma_{32}^2 + \sigma_{42}^2)}, \qquad \Lambda_4 = \frac{b\sigma_{42} - \sigma_{32}c}{2(\sigma_{32}^2 + \sigma_{42}^2)}, \tag{14}$$

where

$$c = \sqrt{4(\sigma_{32}^2 + \sigma_{42}^2)(4a + \sigma_{32}^2 + \sigma_{42}^2) - b^2}.$$

We next introduce the following system of linear PDE's:

$$\begin{aligned}
\frac{\partial U_1}{\partial v} &= U_2 \\
\frac{\partial U_3}{\partial v} &= \frac{\partial U_2}{\partial u} \\
\frac{\partial U_2}{\partial v} &= \frac{1}{U_1}\left(C_3^2 - 2\Lambda_3 C_3 - 2\Lambda_4 C_4 - \sigma_{42}^2 + U_2^2 + U_3^2 - U_1 \frac{\partial U_3}{\partial u}\right) \\
\frac{\partial \eta}{\partial v} &= \frac{2}{U_1^2}(\Lambda_4\sigma_{32} - \Lambda_3\sigma_{42}) \\
\frac{\partial \sigma_{32}}{\partial v} &= 2\frac{\partial \Lambda_3}{\partial u} - \frac{\partial C_3}{\partial u} - 2\eta\Lambda_4 + \eta C_4 + \frac{2}{U_1}U_2(C_3 - \Lambda_3) \\
\frac{\partial C_3}{\partial v} &= -\frac{\partial \sigma_{32}}{\partial u} + \eta\sigma_{42} + \frac{2}{U_1}U_2(C_3 - \Lambda_3) \\
\frac{\partial \sigma_{42}}{\partial v} &= 2\frac{\partial \Lambda_4}{\partial u} - \frac{\partial C_4}{\partial u} + 2\eta\Lambda_3 - \eta C_3 + \frac{2}{U_1}U_3(C_4 - \Lambda_4) \\
\frac{\partial C_4}{\partial v} &= -\frac{\partial \sigma_{42}}{\partial u} - \eta\sigma_{32} + \frac{2}{U_1}U_2(C_4 - \Lambda_4),
\end{aligned} \tag{15}$$

with initial conditions

$$\begin{aligned}
&U_1(u, 0) \equiv 1 \\
&U_2(u, 0) \equiv 0 \\
&U_3(u, 0) \equiv 0 \\
&\sigma_{32}(u, 0) \equiv 0 \\
&\sigma_{32}(u, 0) \equiv 16(a(u, 0))^2 + (b(u, 0))^2 + 2.
\end{aligned} \tag{16}$$

For this system to be well defined, we assume that $U_1 > 0$ and that

$$4(\sigma_{32}^2 + A_{42}^2)(4a + A_{32}^2 + A_{42}^2) - (b(u, v))^2 > 0.$$

Then the Cauchy-Kowalewsky theorem [Spi] implies the existence of an analytic solution around $(0, 0)$ for the entire system (15). Note that the chosen initial conditions guarantee that $U_1 > 0$ and that the expression inside the square root which defines $c = c(u,v)$ is positive and analytic in a small neighborhood of $(0,0)$. If we define $\lambda = U_1$, then the first two equations of (15) together with the chosen initial conditions imply that $U_2 = \lambda_v$ and that $U_3 = \lambda_u$. Also, if we rewrite system (15) by making the following substitutions

$$\begin{aligned} \sigma_{31}(u,v) &:= \Lambda_3(u,v,\sigma_{32}(u,v),\sigma_{42}(u,v)) \\ \sigma_{41}(u,v) &:= \Lambda_4(u,v,\sigma_{32}(u,v),\sigma_{42}(u,v)) \\ \tau_3 &:= C_3 - \sigma_{31} \\ \tau_4 &:= C_4 - \sigma_{41}, \end{aligned} \tag{17}$$

then system (15) implies that the structural equations (6) – (8) are satisfied. Moreover by (14) and (17), we have that (13) is satisfied. □

3. Simple umbilic points

Let $f : M \to \mathbb{R}^4$ be a smooth immersion of a surface M, and let $p \in M$ be a umbilic point of f. The point p is called a *simple umbilic point* of f if there are isothermic coordinates $(u,v) : (M,p) \to (\mathbb{R}^2, 0)$ such that the differential equation of the principal lines of f in these coordinates is of the form

$$4a(u,v)(du^2 - dv^2)dudv + b(u,v)(du^4 - 6du^2dv^2 + dv^4) = 0, \tag{18}$$

with $a = a(u,v)$ and $b(u,v)$ real valued functions which are transversal at the origin.

The proposition and lemma of this paragraph state properties of simple umbilic points which will be necessary later on.

Proposition 3.1. *Any smooth immersion $f : M \to \mathbb{R}^4$ of a surface M can be arbitrarily approximated, in the smooth topology, by an immersion $g : M \to \mathbb{R}^4$ such that all of its umbilic points are simple.*

Proof. Up to a small perturbation, f can be assumed to be analytic. Around any given point of M, in local coordinates and by Theorem 2.1, the condition that $\{a = 0\}$ and $\{b = 0\}$ are made up of regular curves which meet each other transversally is open and dense in the smooth topology. Under these conditions, each element of $\{a = 0\} \cap \{b = 0\}$ is a simple umbilic point. From this local fact, by standard arguments of transversality ([M-P]), the result now follows. □

Lemma 3.2. *Let $f : M \to \mathbb{R}^4$ be a smooth immersion of a surface M, and let $p \in M$ be a simple umbilic point of f. There are isothermic coordinates $(u, v) : (M, p) \to (\mathbb{R}^2, 0)$ such that the differential equation of the principal lines of f in these coordinates is of the form*

$$4(Au + Bv + S)(du^2 - dv^2)dudv + (v + R)(du^4 - 6du^2dv^2 + dv^4) = 0 \quad (19)$$

where $A \neq 0$ and B are real numbers, $S = S(u, v)$ and $R = R(u, v)$ are real valued functions satisfying

$$S(0,0) = R(0,0) = \frac{\partial S}{\partial u}(0,0) = \frac{\partial S}{\partial v}(0,0) = \frac{\partial R}{\partial u}(0,0) = \frac{\partial R}{\partial v}(0,0) = 0.$$

Proof. Let $(s, t) : (M, p) \to (\mathbb{R}^2, 0)$ be isothermic coordinates, and let

$$\omega = 4\tilde{a}(s,t)(ds^2 - dt^2)dsdt + \tilde{b}(s,t)(ds^4 - 6ds^2dt^2 + dt^4) \quad (20)$$

be the corresponding differential equation of the principal lines of f (see Theorem 2.1).

Assume that the first jet

$$J_1(\tilde{a}, \tilde{b})(0,0) = (\tilde{A}_{10}s + \tilde{A}_{01}t, \tilde{B}_{10}s + \tilde{B}_{01}t).$$

For $\alpha, \beta \in \mathbb{R}$, with $\alpha^2 + \beta^2 \neq 0$, we consider

$$(s,t) = \phi(u,v) = (\alpha u - \beta v, \beta u + \alpha v).$$

Then

$$\phi^*\omega = 4a(u,v)(du^2 - dv^2)dudv + b(u,v)(du^4 - 6du^2dv^2 + dv^4)$$

where

$$a(u,v) = Au + Bv + R_1(u,v),$$
$$b(u,v) = B_{10}u + B_{01}v + R_2(u,v)$$

and

$$\begin{aligned} B_{10} = {} & 4\alpha^4\tilde{A}_{10}\beta + 4\alpha^3\tilde{A}_{01}\beta^2 - 4\alpha^2\tilde{A}_{10}\beta^3 - 4\alpha\tilde{A}_{01}\beta^4 + \\ & \alpha^4\beta\tilde{B}_{01} - 6\alpha^2\beta^3\tilde{B}_{01} + \beta^5\tilde{B}_{01} + \alpha^5\tilde{B}_{10} - 6\alpha^3\beta^2\tilde{B}_{10} + \alpha\beta^4\tilde{B}_{10} \\ B_{01} = {} & 4\alpha^4\tilde{A}_{01}\beta - 4\alpha^3\tilde{A}_{10}\beta^2 - 4\alpha^2\tilde{A}_{01}\beta^3 + 4\alpha\tilde{A}_{10}\beta^4 + \\ & \alpha^5\tilde{B}_{01} - 6\alpha^3\beta^2\tilde{B}_{01} + \alpha\beta^4\tilde{B}_{01} - \alpha^4\beta\tilde{B}_{10} + 6\alpha^2\beta^3\tilde{B}_{10} - \beta^5\tilde{B}_{10}. \end{aligned}$$

If $\tilde{B}_{10} = 0$ (thus $\tilde{B}_{01} \neq 0$), we set $\beta = 0$ and $\alpha = \frac{1}{\tilde{B}_{01}^{\frac{1}{5}}}$ to obtain $B_{10} = 0$ and $B_{01} = 1$.

If $\tilde{B}_{10} \neq 0$, we set $\alpha = m\beta$ with m a real root of the equation

$$\tilde{B}_{10}x^5 + 2(2\tilde{A}_{01} - 3\tilde{B}_{10})x^4 + 2(2\tilde{A}_{01} - 3\tilde{B}_{10})x^3 - 2(2\tilde{A}_{10} + 3\tilde{B}_{01})x^2 + (\tilde{B}_{10} - 4\tilde{A}_{01})x + \tilde{B}_{01} = 0$$

to obtain $B_{10} = 0$ and hence we are under the condition of the first case.

□

4. The manifold LM and the semi-local vector field $\mathcal{L}'$

We now consider the projective line bundle PM over M: it is defined by the tangent bundle with the zero section 0 removed $(TM \setminus O)$ modulo the identification of two elements (p_1, v_1) and (p_2, v_2), if their first components coincide and their second ones are collinear. We let P denote the projection of PM onto M. In terms of the chart (u, v) with domain U in M, the charts $(u, v; t = du/dv)$ and $(u, v, s = dv/du)$ are defined on $P^{-1}(U)$ and their domains cover this open set.

Consider the surface LM in PM defined by the solutions of equation (3) of Theorem 2.1:

$$\omega = 4a(u,v)(du^2 - dv^2)dudv + b(u,v)(du^4 - 6du^2dv^2 + dv^4) = 0.$$

In the chart $(u, v; s = dv/du)$ of above, LM is written as

$$\mathcal{L}(u,v;s) = 4a(u,v)(1 - s^2)s + b(u,v)(1 - 6s^2 + s^4) = 0,$$

whereas in the chart $(u, v; t = du/dv)$ it is expressed as

$$\mathcal{L}(u,v;t) = 4a(u,v)(t^2 - 1)t + b(u,v)(t^4 - 6t^2 + 1) = 0.$$

It is clear that the surface LM is determined by the principal directions and does not depend on the particular chart used.

Let Sm be the set of umbilic points of the immersion $f : M \to \mathbb{R}^4$. Outside $P^{-1}(Sm)$ we have that LM is a regular submanifold of $P^{-1}(M)$; there it is a 4-fold regular covering of $M \setminus Sm$. In local (u, v) coordinates around $p \in Sm$, as in Theorem 2.1, Sm corresponds to the set $a^{-1}(0) \cap b^{-1}(0)$.

Lemma 4.1. *Let $p \in Sm$. The point p is simple if and only if LM is regular around $P^{-1}(p)$.*

Proof. Assume the notations and conditions of Lemma 3.2. If for some s and $u = 0, v = 0$ we have that

$$\mathcal{L}_u = 4A(1-s^2)s = 0, \quad \mathcal{L}_v = 4B(1-s^2)s + 1 - 6s^2 + s^4 = 0,$$

then we necessarily have that $A = 0$.

Conversely, if $A = 0$, then $\mathcal{L}_u(s) = 0$, for all s. Since $\mathcal{L}_v(0) = 1$ and $\mathcal{L}_v(1) = -4$, there exists s such that $\mathcal{L}_v(s) = 0$. A similar argument works for the t−coordinate which is needed to analyze the point $t = 0$. □

On local u, v and s coordinates (i.e., $t \neq 0$) for a point of PM, we consider the vector field

$$\mathcal{L}' = u'\frac{\partial}{\partial u} + v'\frac{\partial}{\partial v} + s'\frac{\partial}{\partial s}$$

whose components are given by:

$$\begin{aligned} u' &= \tilde{u}(u,v,s) = 4a(u,v)(1-3s^2) + 4b(u,v)s(-3+s^2) \\ v' &= s\tilde{u}(u,v,s) \\ s' &= -[\ \mathcal{L}_u(u,v;s) + s\mathcal{L}_v(u,v;s)]. \end{aligned}$$

A simple calculation shows that $\mathcal{L}'$ is tangent to LM; in the sequel, we only deal with its restriction to LM whence we shall maintain the same notation $\mathcal{L}'$. Its projection $P_*\mathcal{L}'$ only vanishes at the umbilic points Sm. In the complement of Sm, it generates the principal line fields of M: that is, for each non-umbilic (u,v), the four P-preimages (u,v,r_1), (u,v,r_2), (u,v,r_3) and (u,v,r_4) verify that $P_*\mathcal{L}'(u,v,r_i)$ generates the principal line with direction r_i.

If (u,v) are the coordinates of Lemma 3.2, then $(0,0)$ is umbilic and the singularities of $\mathcal{L}'$ are the zeros of s' on the s-axis given by the equation

$$g(s) = -sQ(s) = 0$$

where

$$Q(s) = s^4 - 4Bs^3 - 2(3+2A)s^2 + 4Bs + 1 + 4A.$$

Lemma 4.2. *Consider*

$$\Delta = 16[4(1+B^2)^3 + 24(1+B^2)^2 A + 8(5-B^2)(1+B^2)A^2 + 4(9+B^2)A^3 + (17+B^2)A^4 + 4A^5],$$

as in Theorem 1.1, and the degree-5- polynomial $g(s) = -sQ(s)$. *Then*

(a) $\Delta < 0$ *implies that* $g(s)$ *has three simple roots;*

(b) $\Delta > 0$ *and* $A \neq -1/4$ *imply that* $g(s)$ *has five simple roots.*

Proof. To find the roots of a quartic polynomial and following [B-P], the principal quantities associated to $Q(s)$ are:

$$\Delta = \Delta(A,B),$$
$$H = H(A,B) = (-3 - 3B^2 - 2A)/3,$$
$$N = N(A,B) = -4(2 + 5B^2 + 3B^4 + 2A + 4B^2A + A^2).$$

When $\Delta < 0$, the real roots of $Q(s)$ are exactly two; this proves statement (a), since $Q(0) \neq 0$. In fact, if $0 = Q(0)$, then $A = -1/4$ and$\Delta(-1/4, B) = B^2(125 + 325B^2 + 256B^4) \geq 0$, for all B, which is not possible.

Statement (b) follows from

(b') $\{\Delta > 0\} \subset \{H < 0\} \cap \{N < 0\}$,

since this implies that Q has four real roots ([B-P]) all of which are nonzero by the assumption $A \neq -1/4$. The proof of (b') is done in 1-8 below:

1. The curve $\{H = 0\}$ is the parabola $A = (-3 - 3B^2)/2$, hence H is negative (resp. positive) on the A-axis, for all $A < -3/2$ (resp. $A > -3/2$). See Figure 4.

2. The curve $\{N = 0\}$ is symmetric with respect to the A-axis and has two connected components. Each component looks like a parabola, with one of them contained in the cone $\{(A,B) : A < -2 - \sqrt{15}/2$ and $B > 0\}$. The complement of $\{N = 0\}$ in the (A,B)-plane is made up of three connected components; N is negative in the one containning the origin (see Figure 4). To see this note that if

$$r(y) = 1 + (1+A)^2 + (5+4A)y + 3y^2 = 0,$$

then

$$\{N = 0\} = \{(A, B) : y = B^2 \quad \text{and} \quad r(y) = 0\}.$$

Therefore when $A < -2 - \sqrt{15}/2$ (resp. $A > -2 - \sqrt{15}/2$), we have that $r(y) = 0$ has two positive roots (resp. has no positive roots).

3. The curve $\{\Delta = 0\}$ is symmetric with respect to the A-axis and has three connected components. Each component looks like a parabola, with one of them tangent to $\{A = -1/4\}$ at $(-1/4, 0)$ and contained in $\{A \leq -1/4\}$. Along the A-axis we have that Δ is positive (resp. negative) for $A > -1/4$ (resp. $A < -1/4$). Another component of $\{\Delta = 0\}$ is tangent to $\{A = -27/8\}$ and is contained in the cone $\{(A, B) : A \leq -27/8 \quad \text{and} \quad B > 0\}$. See Figure 4.

In fact, $\Delta(A, B) = f_A(B^2)$ where, for each A, we have that $f_A(x)$ is a cubic polynomial with discriminant

$$\frac{256}{27} A^8 (27 + 8A)^3.$$

For $A > -\frac{27}{8}$, the polynomial $f_A(x)$ has a unique real root which is positive only for $-\frac{27}{8} < A < -\frac{1}{4}$. For $A < -\frac{27}{8}$, the polynomial $f_A(x)$ has three positive real roots.

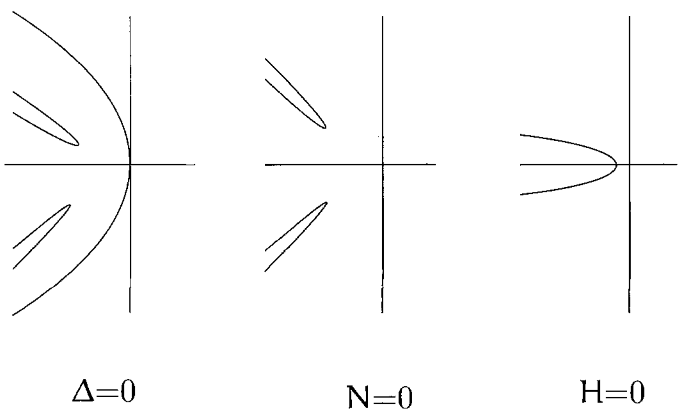

Figure 4.

4. We have that $\{H = 0\} \subset \{\Delta < 0\}$.

In fact, $H = 0$ if and only if $A = (-3 - 3B^2)/2$. Substituting A for this value in Δ, we obtain

$$\Delta = -(1 + B^2)^3(125 - 225B^2 + 162B^4)$$

which is negative for all B.

5. Now $\{\Delta = 0\} \cap \{N = 0\} = \varnothing$.

In effect, considering Δ and N as polynomials in the variable A, their corresponding resultant is the polynomial

$$R = 262144B^4(1 + B^2)^4 R_1$$

where $R_1 = h(B^2)$, with

$$h(x) = -2000 - 776x + 1575x^2 - 648x^3.$$

We have that R vanishes only where $B = 0$ or $R_1 = 0$. When $B = 0$ we have

$$N = -4[\, 1 + (1 + B^2)^2] < 0.$$

Therefore $\{A : \Delta(A, 0) = N(A, 0) = 0\} = \varnothing$. Moreover, since the cubic polynomial $h(x)$ has a unique real root which is negative, $R_1 \neq 0$. and this statement is proved.

6. Next $\{N = 0\} \subset \{\Delta < 0\}$.

In fact, by (5) and since

$$N(-2 - \sqrt{15}/2, \sqrt{1/2 + 1/3\sqrt{15}}) = 0$$

and

$$\Delta(-2 - \sqrt{15}/2, \sqrt{1/2 + 1/3\sqrt{15}}) = (-10821 + 2794\sqrt{15})/9 < 0.$$

7. Also $\{\Delta > 0\} \subset \{H < 0\}$. See Figure 5.
In fact, on the line $A = -27/8$ we have

$$\Delta = 512(-289 + 8B^2)(-125 + 64B^2)^2$$

and

$$H = 5/4 - B^2.$$

Therefore, over this line, $\{\Delta > 0\} \subset \{H < 0\}$ and the result follows from (4).

8. Finally, we have that $\{\Delta > 0\} \subset \{N < 0\}$. See Figure 5.

In fact, for $A = -27/8$ we have

$$N = -(425 - 544B^2 + 192B^4)/16$$

which is negative for all value of B and the result follows from (6).

The proof of Lemma 4.2 is now complete. □

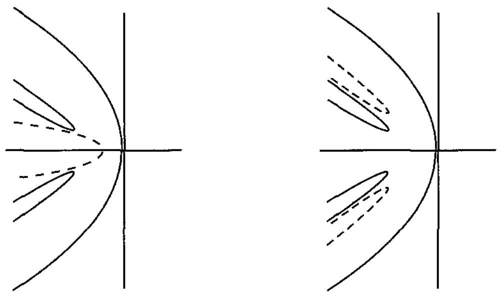

Figure 5.

5. End of the proof of the main result

Lemma 5.1. *Under the generic conditions*

1. $A \neq 0$,
2. $1 + 4A \neq 0$,
3. $\Delta \neq 0$,

the field $\mathcal{L}'$ only has hyperbolic singularities on the s-axis. Moreover:

(a) *Condition* H3 : $\Delta < 0$ *(and so $A < -1/4$) implies that the field $\mathcal{L}'$ has three singular points over the s-axis all of which are saddles.*

(b) *Condition* H4 : $\Delta > 0$, $A < 0$ *and* $A \neq -1/4$ *imply that the field $\mathcal{L}'$ has five singular points over the s-axis, four of which are saddles and the remaining one is a node.*

(c) *Condition* H5 : $A > 0$ *(and so* $\Delta > 0$*) implies that the field $\mathcal{L}'$ has five singular points over the s-axis all of which are saddles.*

Proof. Under condition 1 the curves $a = 0$ and $b = 0$ meet transversally at the origin, and under conditions 2 and 3 the polynomial $g(s)$ only has simple roots. Recall that $g(s) = 0$ is the equation of the singular points of the vector field $\mathcal{L}'$. We observe that if $(0,0,s_0)$ is a singularity of $\mathcal{L}'$, then $\mathcal{L}_v(0,0,s_0) \neq 0$. To see the projection W of our vector field $\mathcal{L}'$ onto the plane u, s, around a singularity of the form $(0,0,s_0)$, from the equation $\mathcal{L}(u,v;s) = 0$, we may write $v = v(u,s)$ (in terms of u and s) and obtain:

$$\begin{aligned} u' &= \tilde{u}(u, v(u,s), s) = 4a(u, v(u,s))(1-3s^2) + 4b(u, v(u,s))s(-3+s^2) \\ &= uh(s) + U(u,s) \\ s' &= -[\ \mathcal{L}_u(u, v(u,s); s) + s\mathcal{L}_v(u, v(u,s); s)] \\ &= g(s) + P(u,s)\,, \end{aligned}$$

with $U(0,s) = \frac{\partial U}{\partial u}(0,s) = 0\,, \quad P(0,s) = \frac{\partial P}{\partial u}(0,s) = 0 \quad$ and

$$h(s) = \frac{4A(1+s^2)^3}{s^4 - 4Bs^3 - 6s^2 + 4Bs + 1}\,.$$

Let $J(s)$ be the determinant of $DW(0,s)$; then:

$$J(0) = -4A(1+4A)$$

and, if $s \neq 0$ is a root of the polynomial g,

$$J(s) = -\frac{(1+s^2)^3}{1-s^2}g'(s)$$

where $g'(s)$ is the derivative of g respect to s.

Conditions 1, 2 and 3 determine seven open regions in the plane A, B : $Z_1, Z_2, \cdots, Z_7$ (see Fig.6). Region Z_7 corresponds to $\Delta < 0$, hence we have three singular points $(0,0,s_i)$, $i = 1,2,3$, and, since $s_1 < -1 < s_2 = 0 < 1 < s_3$, they are hyperbolic saddles. The other regions correspond to $\Delta > 0$ and we therefore have five singular points $(0,0,s_i)$, $i = 1, \cdots, 5$; the relative positions of these points with respect to the points $s = \pm 1$ and the origin as well as their topological type are shown in Table 1, where S (resp. N) stands for saddle point (resp. node) of the vector field W.

region	relative position	topological type
Z_1	$p_1 < -1 < p_2 < p_3 = 0 < p_4 < 1 < p_5$	S S S S S
Z_2	$p_1 < -1 < p_2 < p_3 = 0 < p_4 < 1 < p_5$	S S N S S
Z_3	$p_1 < -1 < p_2 < p_3 < p_4 = 0 < 1 < p_5$	S S N S S
Z_4	$p_1 < -1 < p_2 = 0 < p_3 < p_4 < 1 < p_5$	S S N S S
Z_5	$p_1 < p_2 < p_3 < -1 < p_4 = 0 < 1 < p_5$	S N S S S
Z_6	$p_1 < -1 < p_2 = 0 < 1 < p_3 < p_4 < p_5$	S S S N S

Table 1

The proof of the lemma is now complete. □

Remark 5.2. Under conditions of previous lemma, it follows from its proof that Condition H4 is satisfied if and only if, up to a rotation of the u, v-plane, $-1/4 < A < 0$ (This condition already implies $\Delta > 0$).

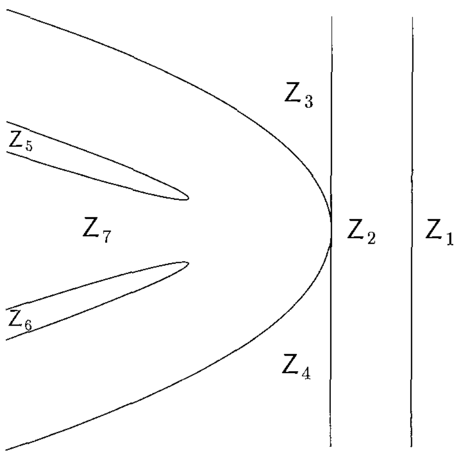

Figure 6.

Proof of Theorem 1.1. It follows from the previous lemma. □

If we denote the set of smooth immersions $f : M \to \mathbb{R}^4$ endowed with the C^∞-topology by $\mathcal{I}_4(M)$, our results may then be summarized in the following theorem.

Theorem 5.3. *The set of smooth immersions $f : M \to \mathbb{R}^4$, such that every umbilic point is locally topologically stable, is open and dense in*

$\mathcal{I}_4(M)$.

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